How to check an iterative scheme is convergent.

Method 1: Check the spectral radius of M.

Here, $x^{k+1} = M x^k + f$.

Example: consider the linear system $Ax = b$,

where $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

Whether the Jacobi method converges?

$A = D + (A - D)$, $\quad D x^{k+1} = (D - A) x^k + b$

$$M = D^{-1}(D-A) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 0 & -2 \\ -3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ -\frac{3}{4} & 0 \end{pmatrix}$$

$$p(\lambda) = \det(M - \lambda I) = \det\left(\begin{pmatrix} -\lambda & -2 \\ -\frac{3}{4} & -\lambda \end{pmatrix}\right) = \lambda^2 - \frac{3}{2}$$

$p(\lambda) = 0 \Rightarrow \lambda = \pm \frac{\sqrt{6}}{2}$

So, Jacobi method does not converge.

Whether Gauss-Seidel converges?

$$M = \frac{1}{4} \begin{pmatrix} 4 & 0 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 0 & \frac{3}{2} \end{pmatrix}.$$

$$p(\lambda) = \det(M - \lambda I) = \lambda\left(\lambda - \frac{3}{2}\right) = \lambda^2 - \frac{3}{2}\lambda$$

$p(\lambda) = 0 \Rightarrow \lambda = 0$ or $\frac{3}{2}$. $\Rightarrow$ Gauss-Seidel method does not work.

Remark: Checking whether the spectral radius is less than 1 can be very expensive because we need some iterative schemes to compute the eigenvalues of M.

Method 2: Check whether A is SDP (strictly diagonally dominant).
(Here, $Ax=b$)

Example: $A = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$. $(|3|>|2|, |4|>|1|)$

Example: Jacobi converges but A is not SDD.

$A = \begin{pmatrix} -3 & 3 & -6 \\ -4 & 7 & -8 \\ 5 & 7 & -9 \end{pmatrix}$, A is not SDD.

However, $M = D^{-1}(D-A) = \begin{pmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{7} & 0 \\ 0 & 0 & -\frac{1}{9} \end{pmatrix} \begin{pmatrix} 0 & -3 & 6 \\ 4 & 0 & 8 \\ -5 & -7 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & 1 & -2 \\ \frac{4}{7} & 0 & \frac{8}{7} \\ \frac{5}{9} & \frac{7}{9} & 0 \end{pmatrix}$

$p(\lambda) = \det(M - \lambda I) = \lambda^3 + \frac{22}{63}\lambda - \frac{16}{63} = 0$

$\Rightarrow \lambda_1 \approx -0.813, \ \lambda_2 \approx 0.4067 - 0.3833i \approx \overline{\lambda_3}$.

$\Rightarrow$ Jacobi method converges.

Example: Gauss Seidel converges but A is not SDD.

$$A = \begin{pmatrix} 4 & 1 & 1 \\ 2 & -9 & 0 \\ 0 & -8 & -6 \end{pmatrix}, \quad A \text{ is not SDD}$$

$$M = \begin{pmatrix} 4 & 0 & 0 \\ 2 & -9 & 0 \\ 0 & -8 & -6 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{-1}{4} & \frac{-1}{4} \\ 0 & -\frac{1}{18} & -\frac{1}{18} \\ 0 & \frac{2}{27} & \frac{2}{27} \end{pmatrix}$$

$$p(\lambda) = \det(M - \lambda I) = 0 \Rightarrow \lambda_1 = 0, \ \lambda_2 = 0, \ \lambda_3 = \frac{1}{54}$$

$\Rightarrow$ Gauss-Seidel method converges.

Method 3: Use Householder-John theorem.

Thm: If $A, B \in \mathbb{R}^{n \times n}$ and both A and $A - B - B^T$ are symmetric and positive definite. Then, the spectral radius of $(A-B)^{-1}B$ is strictly < 1.

Rmk: For the linear system $Nx = b$.

For the Jacobi method, $x^{k+1} = Mx^k + b$, where $M = D^{-1}(D-N)$.

If we let $A = 2D - N$, and $B = D - N$.

Then, $(A-B)^{-1}B = D^{-1}(D-N)$

So, if A and $A - B - B^T$ is symmetric and positive definite, then the Jacobi method converges.

For the Gauss-Seidel method,

let $A = 2D - N$, $B = L$ or U,

then we can check the convergence of Gauss-Seidel.

Proof of thm:

Let λ be an eigenvalue of $-(A-B)^{-1}B$ with one corresponding eigenvector v. (Here, λ can be complex).

Then, $-(A-B)^{-1}B v = \lambda v \Rightarrow -Bv = \lambda(A-B)v$.

$$\Rightarrow -v^* B v = \lambda v^* A v - \lambda v^* B v$$

$(v^* = \bar{v}^T)$

$$\Rightarrow v^* B v = \frac{\lambda}{\lambda - 1} v^* A v.$$

$$\Rightarrow v^* B^T v = (v^* B v)^* = \left(\frac{\lambda}{\lambda-1} v^* A v\right)^* = \frac{\bar{\lambda}}{\bar{\lambda}-1} v^* A v.$$

Since $A - B - B^T$ is positive definite,

$$0 < v^*(A - B - B^T)v$$

$$= v^* A v - \frac{\lambda}{\lambda-1} v^* A v - \frac{\bar{\lambda}}{\bar{\lambda}-1} v^* A v$$

$$= \frac{1 - |\lambda|^2}{|1-\lambda|^2} \underbrace{v^* A v}_{>0}.$$

$$\Rightarrow |\lambda| < 1.$$

Since λ is arbitrary, $(A-B)^{-1}B$ has spectral radius < 1.